Section 3.1 – Extrema on an Interval

Objectives:

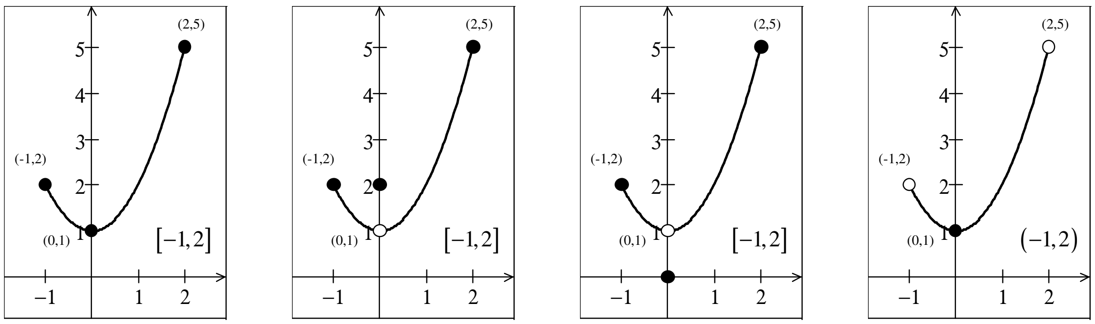
1. Understand the definition of extrema of a function on an interval.
2. Understand the definition of relative extrema of a function on an open interval.
3. Find extrema on a closed interval.

I. Extrema of a Function

A. Absolute Extrema–

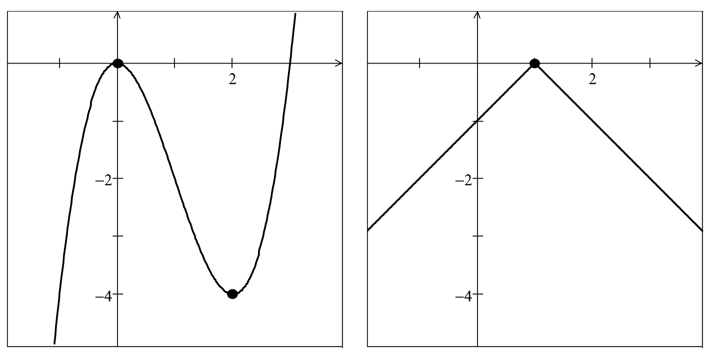
Theorem: The Extreme Value Theorem – If *f* is ____________ on a closed interval [*a*,*b*], then *f* must have both a ____________ and a _____________ on the interval.

B. Relative Extrema (or local extrema) –



II. Relative Extrema and Critical Numbers

A. Def – Let *f* be defined at *c*. If *f* is not _______________ at *c* or $f'(c)=0$, then *c* is a _____________ number of *f*.



B. Relative Extrema occur only at Critical Numbers, if they occur at all.

Theorem: If *f* has a relative extreme at *x* = *c*, then *c* is a critical number of *f*.

C. Examples: Find the extrema on the given closed intervals

1. $y=3x^4-4x^3$ on the interval $[-1,2]$

2. $f(x)=2x-3x^{2/3}+2$ on the interval [-1,3]

3. $f(x)=2\sin x-\cos 2x$ on the interval $[0,2\pi]$

Homework:

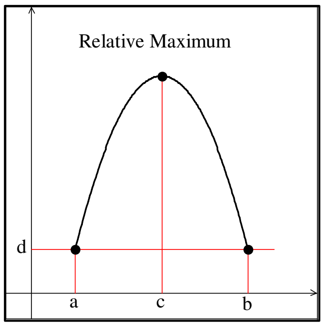
p.169 – 1-10 all, 11, 13, 15, 17, 19, 23, 27, 29, 33, 41, 43, 54-60 all, 61, 65-68 all

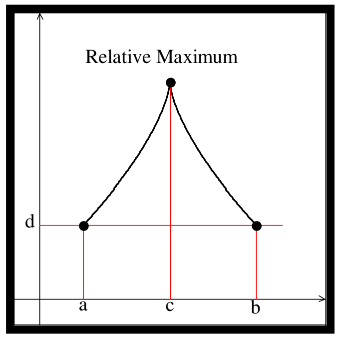
Section 3.2 – Rolle's Theorem and The Mean Value Theorem

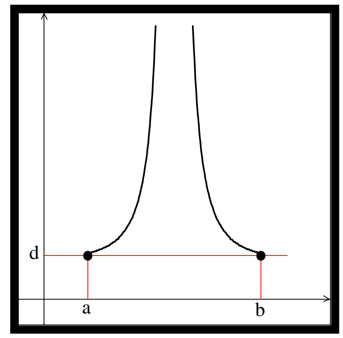
Objectives:
1. Understand and use Rolle's Theorem.
2. Understand and use the Mean Value Theorem.

I. Rolle's Theorem

A. Rolle's Theorem – Let *f* be ______________ on the closed interval [*a*,*b*] and ______________ on the open interval (*a*,*b*). If $f(a) = f(b)$, then there is ___ ________ _____ number *c* in (*a*,*b*) such that $f'(c) = 0$







B. Example:
Does Rolle's Theorem apply to the following functions on the given interval?
If so, find all the c values such that $f'(c) = 0$

1. $f(x) = \frac{x^3}{3} - 3x$ on the interval $(-3,3)$

2. $f(x) = \frac{x-1}{x}$ on the interval $(-1,5)$

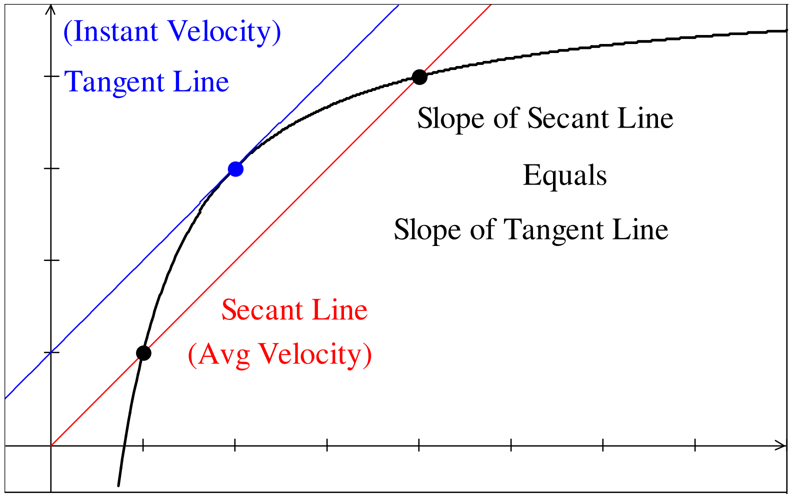
3. $f(x) = |x-2|$ on the interval $(0,5)$

II. The Mean Value Theorem for Derivatives.

A. Theorem:

1. The Mean Value Theorem for Derivatives – If *f* is ___________ on a closed interval [*a*,*b*] and _____________ on the open interval (*a*,*b*), then there ___________ a number *c* such that $f'(c) = \frac{\qquad}{\qquad}$

2. (i.e.) MVT – That the _____________ slope at a _________ in the interval must equal the ___________ slope over the given ____________.



B. Example

1. Does the MVT apply to the following function on the given interval? If so, find c such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

   $f(x) = 5 - \dfrac{4}{x}$ on the interval $(1, 4)$

2. Two stationary patrol cars equipped with radar are 5 miles apart on a highway. As a vehicle passes the first patrol car, its speed is clocked at 70 mph. Four minute later, when it crosses the second patrol car it is clocked at 65 mph. Did the vehicle exceed the 70 mph speed limit during those 4 minutes?

Homework:
p. 172 – 1, 2, 3, 5, 11, 13, 15, 19, 21, 23, 25, 29, 31, 33, 37, 39, 43, 49, 50, 51, 53, 55-57 all, 71, 73-80 all

Section 3.3 – Increasing and Decreasing Functions and the First Derivative

Objectives:

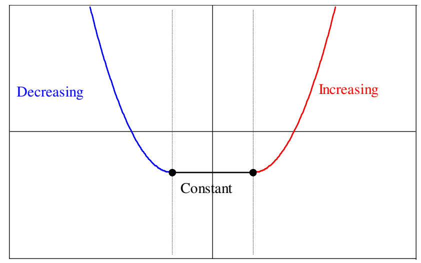
1. To determine the intervals where a function increases or decreases.

I. Increasing and Decreasing Functions

A. Tests

Let *f* be a function that is _______________ on the interval [*a*,*b*] and _____________ on the open interval (*a*,*b*).

1. If $f'(x) > 0$, then *f* is ____________.
2. If $f'(x) < 0$, then *f* is ____________.
3. If $f'(x) = 0$, then *f* is ____________.



B. Example:

1. Find the intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

2. Method
   a) Locate CP's
   b) Pick a test value.
   c) Make conclusion.

II. The First Derivative Test

A. Test:

1. If f' switches from $\Rightarrow$ CP is a maximum.
2. If f' switches from $\Rightarrow$ CP is a minimum.
3. If f' does not switch signs at CP $\Rightarrow$ CP is neither, but a Point of Inflection.

B. Examples

Find the relative extrema for the following functions on the given interval and verify on your calculator

1. $f(x) = \frac{1}{2}x - \sin x$ on the interval $(0, 2\pi)$

2. $f(x) = (x^3 - 4)^{\frac{2}{3}}$

3. $f(x) = \frac{x^4 + 1}{x^2}$

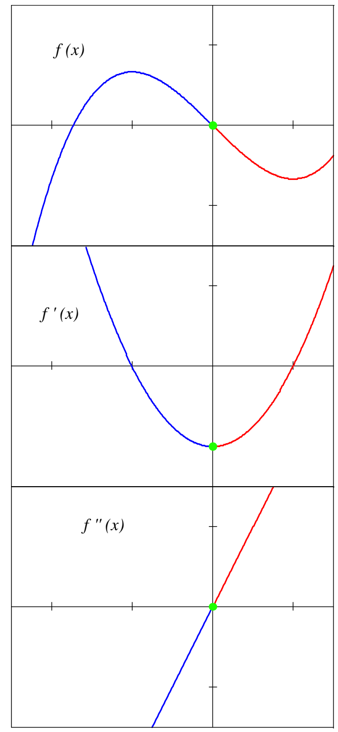
Homework:

p. 186 – 1, 4, 6, 13, 15, 21, 26, 29, 31, 33, 39, 43, 59-70 all, 71-77 all, (79, 80) b and c only

Section 3.4 – Concavity and Second Derivative Test

Objectives:

1. Determine intervals on which a function is concave upward or concave downward.
2. Find any points of inflection of the graph of a function.
3. Apply the Second Derivative Test to find relative extrema of a function.



I. Concavity

A. Definition

Let f be ______________ on open interval I. The graph of f is concave ______________ on I if f' is ____________ on the interval and concave ______________ of I if f' is ______________ on the interval.

Explain how this works –

B. Test for Concavity

1. If $y'' < 0$, then concave _______________.
2. If $y'' > 0$, then concave _______________.

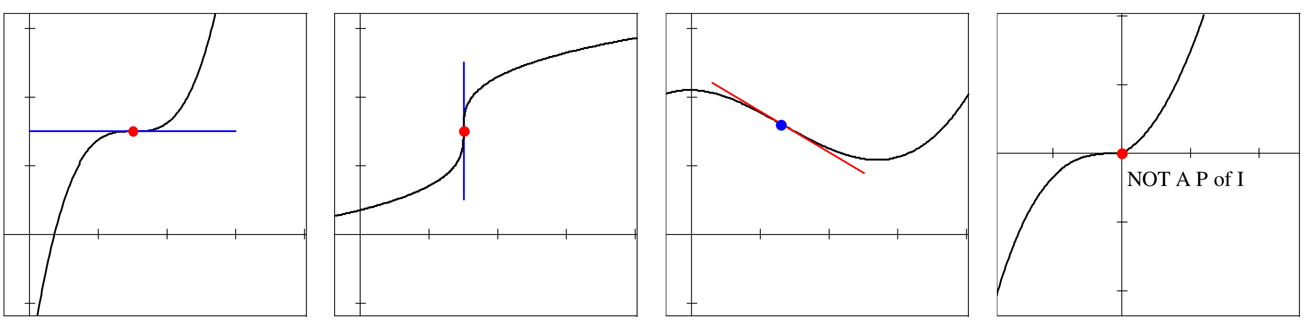
C. Example:

$y = x^3 - 2x^2 + x - 30$

II. Points of Inflection

A. Point of Inflection

1. A point where the _____________ changes.
2. Requires that the tangent line exists at that point.
3. The graph crosses its tangent line at a P of I.



4. Can only occur at $(c, f(c))$ when $f''(c) = 0$ or when $f''(c)$ does not exist.

B. Example

Find the points of inflection, if they exist, for the following functions.

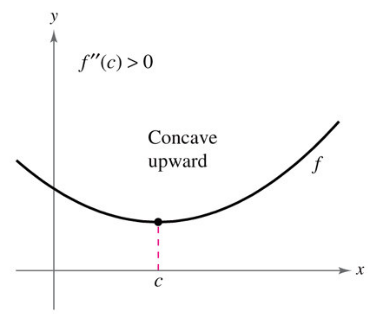
1. $f(x) = x^3 - 4x$ 2. $f(x) = x^4 - 3$

III. Second Derivative Test

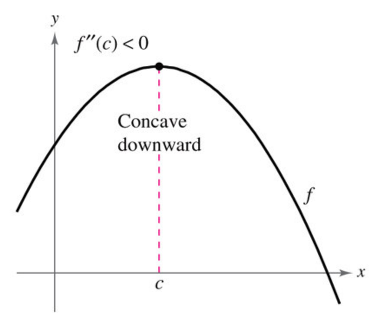
A. Test

Let *f* be a function such that $f'(c) = 0$ and the second derivative of *f* exists on the open interval containing *c*.

1. If $f''(c) > 0$, then $f(c)$ is a relative _______________.
2. If $f''(c) < 0$, then $f(c)$ is a relative _______________.
3. If $f''(c) = 0$, then the test fails. Use First Derivative Test.



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

B. Example

Use the 2nd Derivative Test to find the relative extrema.

1. $f(x) = -3x^5 + 5x^3$

2. $f(x) = \dfrac{x^4 + 1}{x^2}$

Homework:

p. 195 – 1-4 all, 5, 11, 15, 21, 24, 27, 34, 39, 43, 49, 51, 57-64 all, 65, 67, 70, 73, 91-94 all

Section 3.5 – Limits at infinity

Objectives:

1. To find the value of a function (end behavior) when *x* approaches infinity.
2. To demonstrate what it means for a function to have a limit as *x* approaches infinity.
3. To find horizontal asymptotes.

I. Limits at Infinity (End Behavior)

Suppose you were to sketch the graph of the function *f* given by $f(x)=\dfrac{3x^2}{x^2+1}$,but you needed to know what was happening to the graph at the left most and right most points. To get an idea of the end behavior we will develop three methods.

A. Numerically

x decreases without bound. | *x* increases without bound.

x	$-\infty \leftarrow$	-100	-100	-10	-1	0	1	10	100	$\rightarrow \infty$
f(x)						0				

f(x) approaches _______. | *f(x)* approaches _______.

B. Graphically

When you graph the function on your calculator, what does it appear to be doing?

C. Analytically

Rule: $\lim\limits_{x\to\pm\infty}\dfrac{c}{x^n}=$ _______

D. Examples:

1. $\lim\limits_{x\to\infty}\dfrac{3x^4-2x^3+3x^2-5x+6}{3x^4}$

2. $\lim\limits_{x\to-\infty}\dfrac{2x^5-3x^2+5}{9x^6-x^3}$

Short Cuts
(From Lab #4)

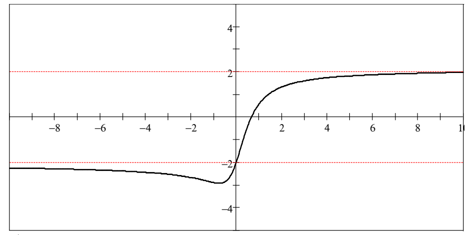
1.

2.

3.

E. Exception to the Rule – Functions that are even indexed radicals

1. $\lim_{x \to \infty} \frac{3x-2}{\sqrt{2x^2+1}}$



2. $\lim_{x \to -\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

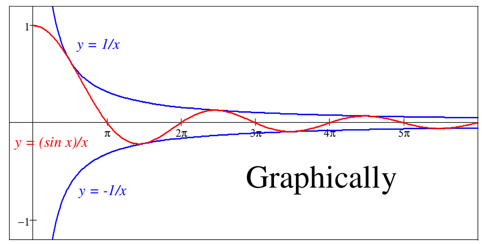
A problem occurs when we divide inside the radical in the denominator by x^2. Why? How can we take care of it?

F. More Examples

1. $\lim_{x \to \infty} \sin x$

2. $\lim_{x \to \infty} \frac{\sin x}{x}$

Algebraically:



Graphically

II. Horizontal Asymptotes

A. If $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$, then $y = b$ is the horizontal asymptote.

B. Examples: Find the horizontal asymptotes, if any.

1. $f(x) = \frac{-x}{x+4}$

2. $f(x) = \frac{x-3}{x^2+2x+4}$

3. $f(x) = \frac{x^2-4}{x^3-2x^2-9x+18}$

III. Infinite Limits at Infinity

Examples:

1. $\lim_{x \to \infty} \frac{2x^2-4x}{x+1}$

2. $\lim_{x \to -\infty} \frac{2x^2-4x}{x+1}$

Homework:

Day 1: p. 205 – 1-6 all, 13, 21, 23, 25, 29, 31, 38, 55, 107, 108

Day 2: Calculator Lab #8

Day 3: p. 215 – 1-4 all, 5-19 odds, 61-64 all, 67, 71

Section 3.7 – Optimization Problems

Objectives:

1. Solve applied minimum and maximum problems.

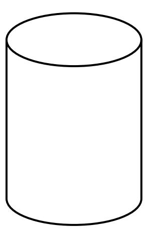
I. Optimization Problems

Guidelines for Solving Applied Minimum and Maximum Problems

1. Identify all *given* quantities and quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the front cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

A. Example

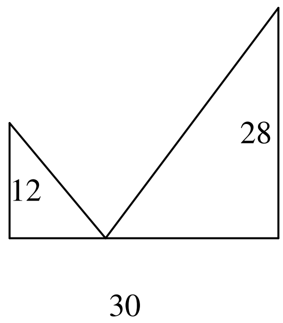
1. You have been asked to design a one-liter oilcan shaped like a right circular cylinder. What dimensions will use the least material? (1L = 1000 cubic cm)



Main Eq:

Secondary Eq:

2. Two posts, one 12 feet high and the other 28 feet high, stands 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least wire?



Primary Eq:

Secondary Eq:

Homework: p. 223 – 17, 21, 23, 26, 43, 49, 56

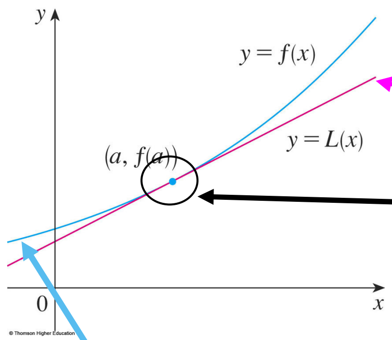
Section 3.9 – Differentials

Objectives:

1. Understand the concept of a tangent line approximation.
2. Compare the value of the differential, dy, with the actual change in y, Δy.
3. Estimate a propagated error using a differential.
4. Find the differential of a function using differentiation formulas.

I. Tangent Line Approximation (Linear Approximation)

A.



The equation of this tangent line is $y - f(a) = f'(a)(x - a)$

By zooming in toward a point on the graph of a differentiable function, we notice that the graph looks more and more like its tangent line.

The approximation
$$f(x) \approx f(a) + f'(a)(x - a)$$
is called the linear approximation or tangent line approximation of f at a.

B. Example

1. Consider the curve defined by $-8x^2 + 5xy + y^3 = -149$

   a) Find $\dfrac{dy}{dx}$.

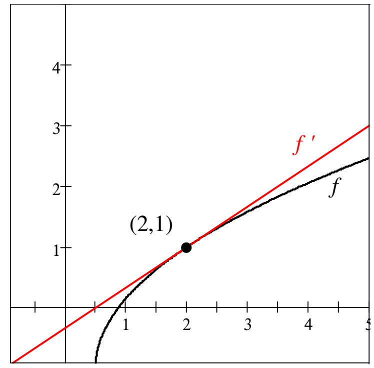
   b) Write an equation for the line tangent to the curve at the point (4,-1)

   c) There is a number k so that the point (4.2, k) is on the curve. Using the tangent line found in part (b), approximate the value of k.

   d) Find the actual value of k on the curve. What was the percent of error?

2. Use the given graph to approximate
   a) $f(1.9)$

   b) $f(2.04)$



II. Differentials

Examples

Function	Derivative	Differential
1. $y = x^2$		
2. $y = 2\sin x$		
3. $y = x\cos x$		
4. $y = \dfrac{1}{x}$		

III. Error Propagation

A. **Physicists and engineers tend to make liberal use of the approximation of Δy by dy. One way this occurs in practice is in the estimation of errors propagated by physical measuring devices.**

x – measured value of a variable — Measured Value
$x + \Delta x$ – the exact value
Δx – the error in measurement — Measurement Error

If the measured value x is used to compute another value $f(x)$, then

$$\underbrace{f(x + \Delta x)}_{\text{Exact Value}} - f(x) = \Delta y \longleftarrow \text{Propagated Error}$$

$$\text{Relativer Error} = \frac{\text{differential}}{\text{actual}} = \frac{dy}{y}$$

B. Example

1. The measured radius of a ball bearing is 0.7 inch. If the measurement is correct within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

$$V = \frac{4}{3}\pi r^3$$

2. Find the relative error and percent of error.

Homework:
p. 240 – 1, 11, 13, 14, 18, 22, 23, 25, 27, 30, 33