

Section 5.1 – The Natural Logarithmic Function and Differentiation

Goals:

1. To define the natural logarithmic function.
2. To review the properties of natural logarithmic functions.
3. To calculate derivatives involving the natural logarithm.

I. Definition and Properties of Natural Logarithms

A. Calculus Definition: $\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$

B. Algebraic Definition: $\ln x = \log_e x$ (Which asks – e to what power equals x ?)

C. Properties:

1. $\ln(1) =$

2. $\ln(ab) =$

3. $\ln\left(\frac{a}{b}\right) =$

4. $\ln(a^n) =$

II. Derivative of $\ln x$

A. $\frac{d}{dx}(\ln u) =$

B. Examples:

1. $\frac{d}{dx}(\ln(x^3 - 2x + 1)) =$

3. $\frac{d}{dx}\left[\ln\left(\frac{2x}{x-3}\right)\right] =$

2. $\frac{d}{dx}[\ln(x-3)^3] =$

4. $\frac{d}{dx}[\ln\sqrt{x+5}] =$

5. $\frac{d}{dx}[\ln((\cos x)(\sin x))] =$

6. Find the relative extrema of $y = \ln(x^2 + 2x + 5)$

Homework: p. 321 – 17-34 all, 41, 45, 49, 50, 61, 59, 59, 70, 73, 77, 89, 100, 105, 106

Section 5.2 – The Natural Logarithmic Function and Integration

Goals:

1. To perform logarithmic integration.
2. To use integration techniques involving natural logarithms for missing trig functions.

I. Logarithmic Integration

A. Log rule for integration: $\int \frac{1}{u} du =$

B. Examples:

1. $\int \frac{5}{x} dx =$

3. $\int \frac{\sec^2 x}{\tan x} dx =$

2. $\int \frac{2x}{x^2 + 3} dx =$

4. $\int_0^3 \frac{x}{x^2 + 1} dx =$

C. Disguised Log Rules for Integration

1. Numerator equal to or greater than denominator (Method: Divide first)

$$\int \frac{x^2 + x + 1}{x^2 + 1} dx =$$

2. u -substitution with Log Rule

Solve the differential equation: $\frac{dy}{dx} = \frac{1}{x \cdot \ln x}$

3. Change of variables with the Log Rule

$$\int \frac{2x}{(x+1)^2} dx =$$

II. Integrals of Trigonometric Function

A. New Integral Rules for Trig Functions

1. Find: $\int (\tan x) dx =$

2. $\int (\cot u) dx =$

3. $\int (\sec u) du =$

4. $\int (\csc u) du =$

B. Examples

1. $\int_0^{\frac{\pi}{4}} (\sqrt{1 + \tan^2 x}) dx =$

2. The electronic force E of a particular electrical circuit is given by $E = 3 \sin 2t$, where E is measured in volts and t is measured in seconds. Find the average value of E as t ranges from 0 to 0.5 seconds.

Homework: p. 330 – 1,5,7,9,13,17,21,17,27,26,35,39,41,42,45,61-64 all,67,77,78,83,87-90 all

Section 5.3 – Inverse Functions

Goals:

1. To find inverse functions.
2. To prove the existence of an inverse function.
3. To find the derivative of an inverse function.

I. Inverses

A. What is the inverse of: $f : \{(1, 4), (2, 5), (3, 6), (4, 7)\}$

B. Definition of an Inverse Function

A function g is the inverse of f , if $f(g(x)) = g(f(x)) = x$ for each x in the domain of g and f . The function g is denoted by f^{-1} (read “ f inverse”).

C. Examples:

1. Find the inverses of :

a) $f(x) = 3x + 2$

b) $f(x) = 3x^2 - 1$

2. Prove that f and g are inverses: $f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\frac{x+1}{2}}$

D. Graphically inverses are reflections over the line:

II. Existence of Inverse Functions

A. To Prove the existence of an Inverse Function

1. A function has an _____ iff it is _____-to-_____.

2. If f is strictly *_____ on its entire domain, then it is one-to-one and therefore has an inverse.

*Note: To prove _____ you must prove that there are not two $f(x)$'s equal on the entire domain. (To do this you must prove that the entire function is either always _____ or always _____.)

B. Examples: Prove that the following either have an inverse function or do not have an inverse.

1. $f(x) = x^3 + x - 1$

2. $f(x) = x^3 - x + 1$

III. Derivative of Inverse Functions

A. Continuity and Differentiability

1. If f is continuous on the domain, then f^{-1} is continuous on the domain.
2. If f is increasing on the domain, then f^{-1} is increasing on the domain.
3. If f is decreasing on the domain, then f^{-1} is decreasing on the domain.
4. If f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

B. The derivative of an Inverse: $g'(x) = \frac{1}{f'(g(x))}$, if $f'(g(x)) \neq 0$

1. If $f(x) = \frac{1}{4}x^3 + x - 1$ find $(f^{-1})'(3)$.

2. If $f(x) = x^2$, $x > 0$, find the slope of f at the point (3,9).
Find the slope of f^{-1} at the point (9,3).

Note: Graphs of inverse function have _____ slopes.

Homework: p. 338 – 5, 9-12 all, 13-21 odds, 25, 27, 29, 39, 43, 49, 63, 71-75 all, 79, 81, 87, 107

Section 5.4 – Exponential Functions: Differentiation and Integration

Goals:

1. To define the natural number.
2. To review the properties of exponents.
3. To find the derivative and integral of the exponential function.

I. The Natural Exponential Function

A. Definition of the Natural Exponential Function

If $f(x) = \ln x$, then $f^{-1}(x) = e^x$ or

B. Properties

1. Equivalent Relation: $a = e^b \Leftrightarrow \ln a = b$
2. $\ln e^x = x$, for all x .
3. $e^{\ln x} = x$, for $x > 0$

C. Simplifying and Solving Exponential and Logarithmic Functions

1. Simplify: $\ln \sqrt{e} =$
2. Simplify: $\ln e^{\sin x} =$
3. Simplify: $e^{3 \ln 2} =$
4. Solve: $7 = e^{x+1}$
5. Solve: $\ln(2x-3) = 5$

D. Other Properties

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$
2. The function $f(x) = e^x$ is continuous, increasing, one-to-one on the entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x \rightarrow -\infty} e^x = 0$ and $\lim_{x \rightarrow \infty} e^x = \infty$

II. Derivatives of Exponential Functions

A. $\frac{d}{dx}(e^u) =$

B. Examples

1. $\frac{d}{dx}[e^{2x-1}] =$
2. $\frac{d}{dx}[e^{-3/x}] =$

3. For 1980 through 1993, the number y of medical doctors in the USA can be modeled by $y = 476,260e^{0.026663t}$ where t represents 1980. At what rate was the number of M.D.'s changing in 1988?

III. Integrals of Exponential Functions

A. $\int e^u du =$

B. Examples

1. $\int e^{3x+1} dx =$

4. $\int_0^{\ln 2} e^{3x} dx =$

2. $\int 5xe^{-x^2} dx =$

5. $\int_0^1 \frac{e^x}{1+e^x} dx =$

3. $\int (\cos x)e^{\sin x} dx$

6. $\int_{-1}^0 e^x \cos(e^x) dx =$

Homework: p. 344 – 3,5-17 odds,23,25-28 all, 38,39,43,47,55,59,69,77,78,87,89,97
101,107,109,113,114,115,121

Section 5.5 – Base other than e and Applications

Goals:

1. To review algebraic properties of exponential functions with a base other than e .
2. To find the derivative and integral of the exponential function with bases other than e .
3. To solve application problems involving exponential functions

I. Bases Other than e

A. Review: $y = a \cdot b^x$

1. a represents
 - a)
 - b)
2. b represents
 - a)
 - b)

B. Properties

1. Equivalent Relation: $a = b^c \Leftrightarrow \log_b a = c$
2. $\log_b b^x = x$, for all x .
3. $b^{\log_b x} = x$, for $x > 0$
4. $\log_b a = \frac{\ln a}{\ln b}$
5. $a^x = e^{x \ln a}$

C. Simplifying and Solving Exponential and Logarithmic Functions

1. Simplify: $\log \sqrt{10} =$
2. Simplify: $\log_3 3^{\sin x} =$
3. Simplify: $5^{3 \log_5 2} =$
4. Solve: $3^x = \frac{1}{81}$
5. Solve: $\log(3x - 2) = 2$
6. The half-life of carbon-14 is about 5730 years. If 1 gram of carbon-14 is present in a sample, how much will be present in 6000 years?

II. Differentiation and Integration

A. Rules:

1. $\frac{d}{dx}(a^u) = u' \cdot a^u \cdot \ln a$
2. $\frac{d}{dx}[\log_a u] = \frac{u'}{u \cdot \ln a}$
3. $\int a^u du = \frac{a^u}{\ln a} + c$

B. Examples

1. Find the derivative of each

a) $y = 2^x$

b) $y = 2^{3x}$

c) $y = \log(\cos x)$

2. $\int 2^x dx =$

3. Find $\frac{dy}{dx}$ if $y = x^x$

III. Applications

A. Recall:

1. Compounding n time in a year: $A = P \left(1 + \frac{r}{n} \right)^{nt}$

2. Compounding continuously: $A = Pe^{rt}$

B. Examples

1. A deposit of \$2500 is made into an account that pays an APR of 5%. Find the balance at the end of 5 years if the interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

2. A bacterial culture is growing according to the logistic growth function

$y = \frac{1.25}{1 + 0.25e^{-0.4t}}$ where y is the weight of the culture in grams and t is time in hours. What is the weight of the culture as t approaches infinity?

Section 5.6 – Differential Equations

Goals:

1. To solve differential equations
2. To apply the law of exponential change to problem solving

I. Differential Equations

Examples

1. Solve: $\frac{dy}{dx} = \frac{2x}{y}$

2. Solve: $\frac{dy}{dx} = ky$

II. Growth and Decay Models

A. Model

1. If the rate of change of a variable y is proportional to the value of y (i.e., $\frac{dy}{dx} = ky$)

than $y = Ce^{kt}$.

2. $y = Ce^{kt}$

a) C –

b) k –

c) If $k > 0$, then

d) If $k < 0$, then

3. $\frac{dy}{dx} = ky$

a) $\frac{dy}{dx}$ –

b) ky –

B. Examples

1. Let $P(t)$ be the number of wolves in a population at time t years, when $t \geq 0$. The population is increasing at a rate proportional to $800 - P(t)$.

a) If $P(0) = 500$, find $P(t)$ in terms of t and k .

b) If $P(2) = 700$, find k .

c) Find $\lim_{t \rightarrow \infty} y$.

2. Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. Given Pu-239 has a half-life of 24,360 years, how long will it take 10 grams to decay to 1 gram?

3. Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after 2 months?

4. Newton's Law of Cooling tells us that the rate at which temperature y changes in a cooling body is proportional to the difference in the temperature in the body and the constant temperature of the surrounding medium.

Let y represent the temperature of an object in a room whose temperature is kept at a constant 60°F . If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80° ?

Homework: p. 366 – 1-6 all, 11-13 all, 15,16, 17,19,21,23,25, 33,45,57,65,74 and p. 377 –25,26, 31,33,35,37,43,45,46,87-100 all,102

Section 5.8 – The Inverse Trig Function

Goals:

1. To review the definitions and properties of the inverse trig functions.
2. To evaluate expressions involving inverse trig functions
3. To differentiate inverse trig functions.

I. Inverse Trig Functions

A. Definitions

1. $y = \sin^{-1} x$ or $y = \arcsin x$
 - a) Domain:
 - b) Range:
2. $y = \cos^{-1} x$ or $y = \arccos x$
 - a) Domain:
 - b) Range:
3. $y = \tan^{-1} x$ or $y = \arctan x$
 - a) Domain:
 - b) Range:

B. Right-triangle Interpretations

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) =$
2. $\arctan\left(-\frac{1}{\sqrt{3}}\right) =$
3. $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) =$
4. Solve: $\arctan(2x-3) = \frac{\pi}{4}$
5. $\cot\left[\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right] =$
6. $\sec\left(\tan^{-1}\frac{x}{3}\right) =$

II. Derivative of Inverse Trig Functions

A. Derivative Rules – see back of front cover (#17 – #22)

B. Examples:

1. $\frac{d}{dx}\left[\sin^{-1}(x^2)\right] =$
2. $\frac{d}{dx}\left[\tan^{-1}\sqrt{x+1}\right] =$
3. $\frac{d}{dx}\left[\sec^{-1}(3x)\right] =$

Section 5.9 – The Inverse Trig Functions and Integration

Goals:

1. To compute integrals leading to inverse trig functions.
2. To review how to complete the square.
3. To review basic integration rules.

I. Integrals Involving Inverse Trig Functions

A. Integral Rules – see back of front cover (#17 – #19)

B. Examples

1. $\int \frac{1}{3\sqrt{1-\frac{x^2}{9}}} dx =$

4. Splitting/Rewriting: $\int \frac{x+2}{\sqrt{4-x^2}} dx =$

2. $\int \frac{x^2}{\sqrt{1-x^6}} dx =$

5. Completing Square: $\int \frac{1}{x^2-4x+7} dx =$

3. Substitution: $\int \frac{1}{\sqrt{e^{2x}-1}} dx =$

6. Complete Square: $\int_{3/2}^{9/4} \frac{1}{\sqrt{3x-x^2}} dx$

II. Which Integration Rule to Use?

A. Example

1. Comparing Integration Problem Set #1

a) $\int \frac{dx}{x\sqrt{x^2-1}}$

b) $\int \frac{xdx}{\sqrt{x^2-1}}$

c) $\int \frac{dx}{\sqrt{x^2-1}}$

2. Comparing Integration Problem Set #2

a) $\int \frac{dx}{x \ln x}$

b) $\int \frac{\ln x}{x} dx$

c) $\int (\ln x) dx$

Homework: p. 390 – 1,7,11,16,21,23,33,39, 47-50 all,51,52,55