

## Section 5.1 – The Natural Logarithmic Function and Differentiation

Goals:

1. Develop and use properties of the natural logarithmic function.
2. Understand the definition of the number  $e$ .
3. Find derivatives of functions involving the natural logarithmic function.

Do Calculator Lab #13

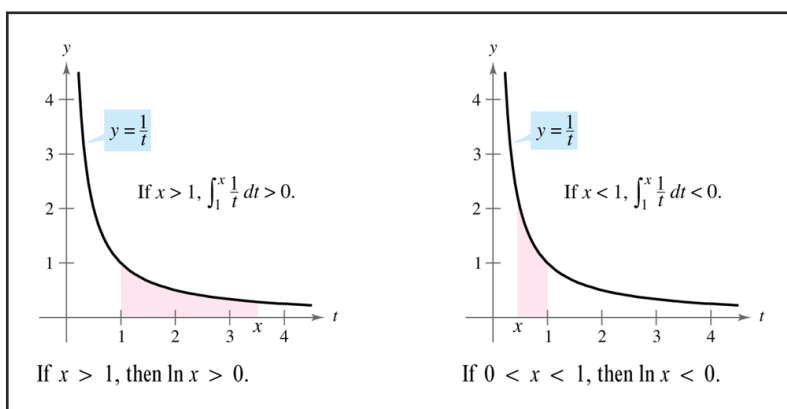
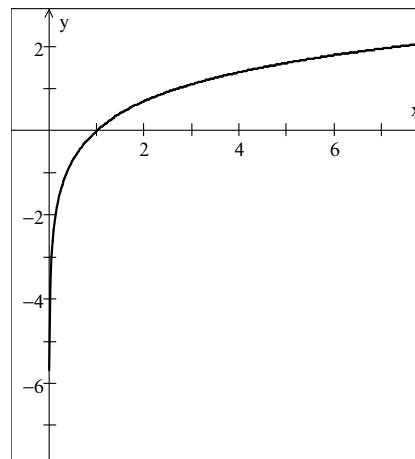
### I. Definition and Properties of Natural Logarithms

A. Algebraic Definition:  $\ln x = \log_e x$

(Which asks –  $e$  to what power equals  $x$ ?)

$$y = \ln x \Leftrightarrow$$

B. Calculus Definition:  $\int_1^x \frac{1}{t} dt =$



### C. Properties:

1.  $\ln(1) =$
2.  $\ln(ab) =$
3.  $\ln\left(\frac{a}{b}\right) =$
4.  $\ln(a^n) =$

#### THEOREM 5.1 Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain is  $(0, \infty)$  and the range is  $(-\infty, \infty)$ .
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Examples

1. Expand the following expressions:

$$\ln \frac{6x}{5} \qquad \ln \sqrt{3x+2}$$

$$\ln \frac{(x^2+3)^2}{x^3\sqrt{x^2+1}}$$

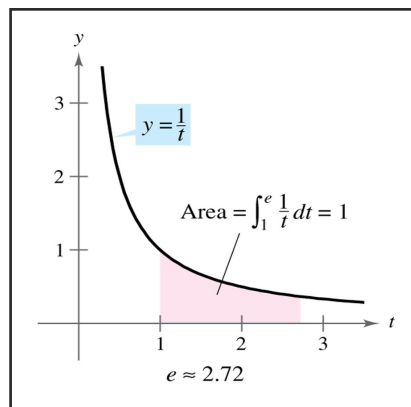
2. Write as a single logarithm:  $\frac{3}{2}[\ln(x^2 + 1) - \ln(x - 1) - \ln(x + 1)]$

II. Definition of  $e$

**Definition of  $e$**

The letter  $e$  denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t} dt = 1.$$



III. Derivative of  $\ln x$

A.  $\frac{d}{dx}(\ln u) =$

B. Examples:

1.  $\frac{d}{dx}(\ln(x^3 - 2x + 1)) =$

3.  $\frac{d}{dx}\left[\ln\left(\frac{2x}{x-3}\right)\right] =$

2.  $\frac{d}{dx}[\ln(x-3)^3] =$

4.  $\frac{d}{dx}[\ln\sqrt{x+5}] =$

5.  $\frac{d}{dx}[\ln((\cos x)(\sin x))] =$

6. Find the relative extrema of  $y = \ln(x^2 + 2x + 5)$

7. Use logarithmic differentiation to find  $\frac{dy}{dx}$ :  $y = \sqrt{x^2(x+1)(x+2)}$

Homework: p. 331 – 7-10 all, 15-18 all, 21-35 odds, 39-42 all, 49-52 all, 57, 63, 69, 75, 77, 80, 82, 83, 85, 88, 93, 94, 101, 103, 11-115 all

## Section 5.2 – The Natural Logarithmic Function and Integration

Goals:

1. To perform logarithmic integration.
2. To use integration techniques involving natural logarithms for missing trig functions.

### I. Logarithmic Integration

A. Log rule for integration:  $\int \frac{1}{u} du =$

B. Examples:

1.  $\int \frac{5}{x} dx =$

3.  $\int \frac{\sec^2 x}{\tan x} dx =$

2.  $\int \frac{2x}{x^2 + 3} dx =$

4.  $\int_0^3 \frac{x}{x^2 + 1} dx =$

### C. Disguised Log Rules for Integration

1. Numerator equal to or greater than denominator (Method: Divide first)

$$\int \frac{x^2 + x + 1}{x^2 + 1} dx =$$

2.  $u$ -substitution with Log Rule

Solve the differential equation:  $\frac{dy}{dx} = \frac{1}{x \cdot \ln x}$

3. Change of variables with the Log Rule

$$\int \frac{2x}{(x+1)^2} dx =$$

## II. Integrals of Trigonometric Function

### A. New Integral Rules for Trig Functions

1. Find:  $\int (\tan x) dx =$

2.  $\int (\cot u) dx =$

3.  $\int (\sec u) du =$

4.  $\int (\csc u) du =$

### B. Examples

1.  $\int_0^{\frac{\pi}{4}} (\sqrt{1 + \tan^2 x}) dx =$

2. The electronic force  $E$  of a particular electrical circuit is given by  $E = 3 \sin 2t$ , where  $E$  is measured in volts and  $t$  is measured in seconds. Find the average value of  $E$  as  $t$  ranges from 0 to 0.5 seconds.

Homework: p. 330 – 1, 5, 7, 9, 13, 17, 21, 27, 31, 35, 39, 41, 42, 45, 49-52 all, 53, 56, 67-70 all, 73, 76, 77, 78, 89, 93, 98, 101, 105-108 all

## Section 5.3 – Inverse Functions

Goals:

1. Verify that one function is the inverse function of another function.
2. Determine whether a function has an inverse function.
3. Find the derivative of an inverse function.

### I. Inverses

A. What is the inverse of:  $f : \{(1, 4), (2, 5), (3, 6), (4, 7)\}$

### B. Definition of an Inverse Function

A function  $g$  is the inverse of  $f$ , if  $f(g(x)) = g(f(x)) = x$  for each  $x$  in the domain of  $g$  and  $f$ . The function  $g$  is denoted by  $f^{-1}$  (read “ $f$  inverse”).

### C. Examples:

1. Find the inverses of :

a)  $f(x) = 3x + 2$

b)  $f(x) = 3x^2 - 1$

2. Prove that  $f$  and  $g$  are inverses:  $f(x) = 2x^3 - 1$  and  $g(x) = \sqrt[3]{\frac{x+1}{2}}$

D. Graphically inverses are reflections over the line:

### II. Existence of Inverse Functions

#### A. To Prove the existence of an Inverse Function

1. A function has an \_\_\_\_\_ iff it is \_\_\_\_\_-to-\_\_\_\_\_.

2. If  $f$  is strictly \*\_\_\_\_\_ on its entire domain, then it is one-to-one and therefore has an inverse.

\*Note: To prove \_\_\_\_\_ you must prove that there are not two  $f(x)$ 's equal on the entire domain. (To do this you must prove that the entire function is either always \_\_\_\_\_ or always \_\_\_\_\_.)

B. Examples: Prove that the following either have an inverse function or do not have an inverse.

1.  $f(x) = x^3 + x - 1$

2.  $f(x) = x^3 - x + 1$

### III. Derivative of Inverse Functions

#### A. Continuity and Differentiability

1. If  $f$  is continuous on the domain, then  $f^{-1}$  is continuous on the domain.
2. If  $f$  is increasing on the domain, then  $f^{-1}$  is increasing on the domain.
3. If  $f$  is decreasing on the domain, then  $f^{-1}$  is decreasing on the domain.
4. If  $f$  is differentiable at  $c$  and  $f'(c) \neq 0$ , then  $f^{-1}$  is differentiable at  $f(c)$ .

B. The derivative of an Inverse:  $g'(x) = \frac{1}{f'(g(x))}$ , if  $f'(g(x)) \neq 0$

1. If  $f(x) = \frac{1}{4}x^3 + x - 1$  find  $(f^{-1})'(3)$ .
  
  
  
  
  
  
  
  
  
  
2. If  $f(x) = x^2$ ,  $x > 0$ , find the slope of  $f$  at the point (3,9).  
Find the slope of  $f^{-1}$  at the point (9,3).

Note: Graphs of inverse function have \_\_\_\_\_ slopes.

3. Some function  $f(x)$  and its derivative  $f'(x)$  are continuous and differentiable for all real numbers, and some of the values for the functions are given in the below table:

$x$	-1	0	1	2	3
$f(x)$	0	2	3	-1	1
$f'(x)$	3	-1	-2	0	1

Based on the information given, answer the following questions:

- (a) Evaluate:  $(f^{-1})'(3)$
- (b) At what value  $c$  is the graph of  $(f^{-1})'(x)$  discontinuous?

Homework: p. 349 – 5, 9-12 all, 13-21 odds, 25, 27, 29, 39, 43, 49, 63, 71, 75, 79, 81

## Section 5.4 – Exponential Functions: Differentiation and Integration

Goals:

1. Develop properties of the natural exponential function.
2. Differentiate natural exponential functions.
3. Integrate natural exponential functions.

### I. The Natural Exponential Function

#### A. Definition of the Natural Exponential Function

If  $f(x) = \ln x$ , then  $f^{-1}(x) = e^x$ . That is, \_\_\_\_\_ iff \_\_\_\_\_.

#### B. Properties

1. The domain of  $f(x) = e^x$  is  $(-\infty, \infty)$ , and the range is  $(0, \infty)$
2. The function  $f(x) = e^x$  is continuous, increasing, one-to-one on the entire domain.
3. The graph of  $f(x) = e^x$  is concave upward on its entire domain.
4.  $\lim_{x \rightarrow -\infty} e^x = 0$  and  $\lim_{x \rightarrow \infty} e^x = \infty$

#### C. Other Properties

1. Equivalent Relation:  $a = e^b \Leftrightarrow \ln a = b$
2.  $\ln e^x = x$ , for all  $x$ .
3.  $e^{\ln x} = x$ , for  $x > 0$

#### D. Simplifying and Solving Exponential and Logarithmic Functions

1. Simplify:  $\ln \sqrt{e} =$
4. Solve:  $7 = e^{x+1}$
2. Simplify:  $\ln e^{\sin x} =$
5. Solve:  $\ln(2x - 3) = 5$
3. Simplify:  $e^{3 \ln 2} =$

### II. Derivatives of Exponential Functions

A.  $\frac{d}{dx}(e^u) =$

#### B. Examples

1.  $\frac{d}{dx}[e^{2x-1}] =$
2.  $\frac{d}{dx}[e^{-3/x}] =$

3. For 1980 through 1993, the number  $y$  of medical doctors in the USA can be modeled by  $y = 476,260e^{0.026663t}$  where  $t$  represents 1980. At what rate was the number of M.D.'s changing in 1988?

End Day 1

III. Integrals of Exponential Functions

A.  $\int e^u du =$

B. Examples

1.  $\int e^{3x+1} dx =$

4.  $\int_0^{\ln 2} e^{3x} dx =$

2.  $\int 5xe^{-x^2} dx =$

5.  $\int_0^1 \frac{e^x}{1+e^x} dx =$

3.  $\int (\cos x)e^{\sin x} dx$

6.  $\int_{-1}^0 e^x \cos(e^x) dx =$

Homework:

Day 1: p. 358 – 3-17 odds, 20, 25-28 all, 38, 39, 43, 47, 55, 59, 69, 79, 91

Day 2: p. 360 – (99-126)/3, 127, 131, 133, 136, 139

## Section 5.5 – Base other than $e$ and Applications

Goals:

1. Define exponential functions that have bases other than  $e$ .
2. Differentiate and integrate exponential functions that have bases other than  $e$ .
3. Use exponential functions to model compound interest and exponential growth.

### I. Bases Other than $e$

A. Review:  $y = a \cdot b^x$

1.  $a$  represents
  - a)
  - b)

2.  $b$  represents
  - a)
  - b)

B. Properties

1. Equivalent Relation:  $a = b^c \Leftrightarrow \log_b a = c$
2.  $\log_b b^x = x$ , for all  $x$ .
3.  $b^{\log_b x} = x$ , for  $x > 0$
4.  $\log_b a = \frac{\ln a}{\ln b}$
5.  $a^x = e^{x \ln a}$

C. Simplifying and Solving Exponential and Logarithmic Functions

1. Simplify:  $\log \sqrt{10} =$
2. Simplify:  $\log_3 3^{\sin x} =$
3. Simplify:  $5^{3 \log_5 2} =$
4. Solve:  $3^x = \frac{1}{81}$
5. Solve:  $\log(3x - 2) = 2$
6. The half-life of carbon-14 is about 5730 years. If 1 gram of carbon-14 is present in a sample, how much will be present in 6000 years?

### II. Differentiation and Integration

A. Rules:

1.  $\frac{d}{dx}(a^u) =$
2.  $\frac{d}{dx}[\log_a u] =$
3.  $\int a^u du =$

### B. Examples

1. Find the derivative of each

a)  $y = 2^x$

b)  $y = 2^{3x}$

c)  $y = \log(\cos x)$

2.  $\int 2^x dx =$

3. Find  $\frac{dy}{dx}$  if  $y = x^x$

### III. Applications

#### A. Recall:

1. Compounding  $n$  time in a year:  $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. Compounding continuously:  $A = Pe^{rt}$

#### B. Examples

1. A deposit of \$2500 is made into an account that pays an APR of 5%. Find the balance at the end of 5 years if the interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

2. A bacterial culture is growing according to the logistic growth function

$$y = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

where  $y$  is the weight of the culture in grams and  $t$  is time in hours.

What is the weight of the culture as  $t$  approaches infinity?

Homework: p. 357 – 1-8 all, 10, 15-18 all, 23, 27, 41, 45, 51, 53, 57, 61, 63, 65, 69, 71, 75-85 odds, 87, 89, 95, 97, 107

## Section 6.2 – Differential Equations

Goals:

1. Use separation of variables to solve a simple differential equation.
2. Use exponential functions to model growth and decay in applied problems.

### I. Differential Equations

Examples

1. Solve:  $y' = \frac{2x}{y}$     Leibniz notation:  $\frac{dy}{dx} = \frac{2x}{y}$

2. Solve:  $\frac{dy}{dx} = ky$

### II. Growth and Decay Models

A. Model

1. If the rate of change of a variable  $y$  is proportional to the value of  $y$  (i.e.,  $\frac{dy}{dx} = ky$ )

than  $y = Ce^{kt}$ .

2.  $y = Ce^{kt}$

a)  $C$  –

b)  $k$  –

(1) If  $k > 0$ , then

(2) If  $k < 0$ , then

3.  $\frac{dy}{dx} = ky$

a)  $\frac{dy}{dx}$  –

b)  $k$  –

B. Examples

1. Let  $P(t)$  be the number of wolves in a population at time  $t$  years, when  $t \geq 0$ . The population is increasing at a rate proportional to  $800 - P(t)$ .

a) If  $P(0) = 500$ , find  $P(t)$  in terms of  $t$  and  $k$ .

b) If  $P(2) = 700$ , find  $k$ .

c) Find  $\lim_{t \rightarrow \infty} P(t)$ .

2. Newton's Law of Cooling tells us that the rate at which temperature  $T$  changes in a cooling body is proportional to the difference in the temperature in the body and the constant temperature of the surrounding medium.

Let  $T$  represent the temperature of an object in a room whose temperature is kept at a constant  $60^\circ\text{F}$ . If the object cools from  $100^\circ$  to  $90^\circ$  in 10 minutes, how much longer will it take for its temperature to decrease to  $80^\circ$ ?

3. Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. Given Pu-239 has a half-life of 24,360 years, how long will it take 10 grams to decay to 1 gram?

4. Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after 2 months?

Homework: p. 420 – (3-27)/3, 33, 37, 43, 48, 49, 55, 71, 73, 75-78 all and  
p. 431 – (4-28)/4, 51, 59-62 all, 90

## Section 5.6 – The Inverse Trig Function

Goals:

1. Develop properties of the six inverse trigonometric functions.
2. Differentiate an inverse trigonometric function.
3. Review the basic differentiation rules for elementary functions.

### I. Inverse Trig Functions

#### A. Definitions

1.  $y = \sin^{-1} x$  or  $y = \arcsin x$ 
  - a) Domain:
  - b) Range:
2.  $y = \cos^{-1} x$  or  $y = \arccos x$ 
  - a) Domain:
  - b) Range:
3.  $y = \tan^{-1} x$  or  $y = \arctan x$ 
  - a) Domain:
  - b) Range:

#### B. Right-triangle Interpretations

- |  |   |
|--|---|
| <ol style="list-style-type: none"> <li>1. <math>\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) =</math></li> <li>2. <math>\arctan\left(-\frac{1}{\sqrt{3}}\right) =</math></li> <li>3. <math>\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) =</math></li> </ol> | <ol style="list-style-type: none"> <li>4. <math>\cot\left[\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right] =</math></li> <li>5. <math>\sec\left(\tan^{-1}\frac{x}{3}\right) =</math></li> <li>6. Solve: <math>\arctan(2x-3) = \frac{\pi}{4}</math></li> </ol> |
|--|---|

### II. Derivative of Inverse Trig Functions

#### A. Derivative Rules –

$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$	$\frac{d}{dx}[\text{arc cot } u] = \frac{-u'}{1+u^2}$
$\frac{d}{dx}[\text{arc sec } u] = \frac{u'}{ u \sqrt{u^2-1}}$	$\frac{d}{dx}[\text{arc csc } u] = \frac{-u'}{ u \sqrt{u^2-1}}$

#### B. Examples:

1.  $\frac{d}{dx}[\sin^{-1}(x^2)] =$
2.  $\frac{d}{dx}[\tan^{-1}\sqrt{x+1}] =$
3.  $\frac{d}{dx}[\sec^{-1}(3x)] =$

Homework: p. 379 – 5-12 all, 17-20 all, 27, 33, 37, 38, 43-49 odds, 58, 63, 68, 73, 77, 78, 81, 97, 104a

## Section 5.9 – The Inverse Trig Functions and Integration

Goals:

1. Integrate functions whose antiderivatives involve inverse trigonometric functions.
2. Use the method of completing the square to integrate a function.
3. Review the basic integration rules involving elementary functions.

### I. Integrals Involving Inverse Trig Functions

A. Integral Rules –

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C \quad \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C$$

B. Examples

1.  $\int \frac{1}{3\sqrt{1 - \frac{x^2}{9}}} dx =$

4. Splitting/Rewriting:  $\int \frac{x+2}{\sqrt{4-x^2}} dx =$

2.  $\int \frac{x^2}{\sqrt{1-x^6}} dx =$

5. Completing Square:  $\int \frac{1}{x^2 - 4x + 7} dx =$

3. Substitution:  $\int \frac{1}{\sqrt{e^{2x} - 1}} dx =$

6. Complete Square:  $\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}} dx$

### II. Which Integration Rule to Use?

A. Example

1. Comparing Integration Problem Set #1

a)  $\int \frac{dx}{x\sqrt{x^2 - 1}}$

b)  $\int \frac{xdx}{\sqrt{x^2 - 1}}$

c)  $\int \frac{dx}{\sqrt{x^2 - 1}}$

2. Comparing Integration Problem Set #2

a)  $\int \frac{dx}{x \ln x}$

b)  $\int \frac{\ln x}{x} dx$

c)  $\int (\ln x) dx$